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Extended multiplet structure in logarithmic conformal field theories

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ABSTRACT: We use the process of quantum hamiltonian reduction of $SU(2)_k$, at rational level k, to study explicitly the correlators of the $h_{1,s}$ fields in the $c_{p,q}$ models. We find from direct calculation of the correlators that we have the possibility of extra, chiral and non-chiral, multiplet structure in the $h_{1,s}$ operators beyond the 'minimal' sector. At the level of the vacuum null vector $h_{1,2p-1} = (p-1)(q-1)$ we find that there can be two extra non-chiral fermionic fields. The extra indicial structure present here permeates throughout the entire theory. In particular we find we have a chiral triplet of fields at $h_{1,4p-1} = (2p-1)(2q-1)$. We conjecture that this triplet algebra may produce a rational extended $c_{p,q}$ model. We also find a doublet of fields at $h_{1,3p-1} = (3p/2-1)(3q/2-1)$. These are chiral fermionic operators if p and q are not both odd and otherwise parafermionic.

KEYWORDS: Field Theories in Lower Dimensions, Conformal and W Symmetry, Integrable Field Theories.

Contents

1. Introduction

The study of conformal invariance in two dimensions has been a fascinating and productive area of research for the last twenty years [1]. There is an interesting class of conformal field theories (CFTs) called logarithmic conformal field theories (LCFTs). In these theories the irreducible primary operators do not close under fusion and indecomposable representations are inevitably generated [2]. The operators in the theory have scaling dimensions that are either degenerate or differ by integers. In these cases it is possible to have a non-trivial Jordan block structure. The association between logarithms in correlation functions and indecomposable representations had been also been noted in the context of super-symmetric WZNW models [3, 4].

LCFTs have emerged in many different areas for example: WZNW models and gravitational dressing [5]–[10], [12]–[14], polymers [15]–[17], disordered systems and the Quantum Hall effect [18]–[33], string theory [34]–[45], 2d turbulence [46]–[50], multi-colour QCD at low-x [51], the abelian sandpile model [52, 53] and the Seiberg-Witten solution of $\mathcal{N} = 2$

SUSY Yang-Mills [54, 55]. Deformed LCFTs, Renormalisation group flows and the c-theorem were discussed in [18, 56, 57]. The holographic relation between logarithmic operators and vacuum instability was considered in [58, 59]. There has also been much interest on LCFTs with a boundary [60]–[63]. For more about the general structure of LCFT see [64]–[67] and references therein. Introductory lecture notes on LCFT and more references can be found in [68]–[71]. A general approach to LCFT via deformations of the operators has been given in [72].

There has also been much work on analysing the general structure and consistency of such models in particular the $c_{p,1}$ models and the special case of $c_{2,1} = -2$ which is by far the best understood [73]–[76]. The key to this understanding is the fact that one may extend to Virasoro algebra by triplets of chiral $h_{3,1} = 2p - 1$ fields [77]. The resulting algebra is sufficient is create a rational LCFT, i.e. one having only a finite number of irreducible and indecomposable representations [78, 79]. The aim of this paper is to show that this extended algebraic structure generalises in a simple way to all $c_{p,q}$ models. We shall leave the question of rationality for future work.

The WZNW model is of great importance in CFT. Correlation functions in such models obey differential, Knizhnik-Zamolodchikov , equations [80] coming from null states in the theory. The solutions to these equations and correlation functions for the integrable sector of the $\mathrm{SU}(2)_k$ model were studied by [81, 82]. In the case of the integrable representations these were previously studied in [83, 84] in which it was found that the rational solutions to the Knizhnik-Zamolodchikov equation were in one-one correspondence with the extensions of the chiral algebra. There is a simple Dotsenko-Fateev integral representation for solutions but these do not converge in many cases beyond the integrable representations. In particular in the cases in which logarithms appear we have to be very careful when analytically continuing the solutions and it is much easier, and more convincing, to solve the equations directly. We shall make use of quantum hamiltonian reduction of $\mathrm{SU}(2)_k$ WZNW models, at rational level k, which gives a very efficient procedure to directly calculate differential equations for the $h_{1,s}$ fields in the $c_{p,q}$ models. By examining the correlation functions in several examples we shall show that there can exist a very simple structure for a certain subset of the $h_{1,s}$ operators.

We find that there is a single rational solution generated by the $h_{1,2p-1} = (p-1)(q-1)$ field corresponding to the vacuum null vector of the irreducible theory. It is well known that decoupling such a null vector gives us a complete description of the 'minimal' $c_{p,q}$ model [85]. However at this conformal weight we find, in addition, two other primary *fermionic* non-chiral operators. This extra structure permeates the model.

We found that there are triplets of chiral bosonic fields at $h_{1,4p-1} = (2p-1)(2q-1)$. These are a natural generalisation of an algebra, generated by the $h_{1,3} = 2q - 1$ fields, that appears in the $c_{1,q}$ models. It has been previously conjectured by M. Flohr [79] that these extended $c_{p,q}$ models should be formally considered as $c_{3p,3q}$ and we conjecture that the algebra of such $h_{1,4p-1}$ fields may yield rational extended $c_{p,q}$ models. We also observed an extra doublet structure of the $h_{1,3p-1} = (3p/2 - 1)(3q/2 - 1)$ fields. If p, q are not both odd then these correspond to chiral fermionic operators otherwise they are parafermionic.

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Recently a particular $\mathrm{SU}(2)_k$ theory at rational level, namely $k = -4/3$, was studied [10] (See also [11] for a study of $\mathrm{SU}(2)_{-1/2}$). It was found that indecomposable representations were created in the fusion of admissable representations and that the theory was *not* rational. On hamiltonian reduction the discrete representations of SU(2) with $2j \in Z^+$, which are different to the admissable representations, produce $h_{1,s}$ fields in the $c_{2,3} = 0$ model. It would be interesting to see if the type of extended algebras studied in this paper could be used to construct rational models of $\widehat{\mathrm{SU}(2)}$ at fractional level.

2. Knizhnik-Zamolodchikov equation

We consider the $\widehat{\mathrm{SU}(2)}$ theory at rational level k. The OPE of the affine Kac-Moody currents is given by:

$$\begin{aligned} J^3(z)J^\pm(w) &\sim \pm\frac{J^\pm(w)}{z-w} \\ J^+(z)J^-(w) &\sim \frac{k}{(z-w)^2} + \frac{2J^3(w)}{z-w} \\ J^3(z)J^3(w) &\sim \frac{k}{2(z-w)^2} \end{aligned} \tag{2.1}$$

We use the standard Sugawara construction for the stress tensor:

$$T = \frac{1}{k+2}\left(\frac{1}{2}J^+J^- + \frac{1}{2}J^-J^+ + J^3J^3\right) \tag{2.2}$$

which yields a theory with central charge:

$$c = \frac{3k}{k+2} \tag{2.3}$$

We consider affine Kac-Moody primary operators having the simple behaviour:

$$J^a(z)\phi_j(w) \sim \frac{t^a\phi_j(w)}{z-w} \tag{2.4}$$

where t^a is a spin j matrix representation of SU(2). We also have affine Virasoro null vectors following from (2.2):

$$|\chi\rangle = \left(L_{-1} - \frac{1}{k+2}\eta_{ab}J^a_{-1}J^b_0\right)|\phi\rangle \tag{2.5}$$

Inserting these null vectors into correlation functions of affine Kac-Moody primaries one can show that they must satisfy a set of partial differential equations known as Knizhnik-Zamolodchikov equations [80]:

$$\left[(k+2)\frac{\partial}{\partial z_i} + \sum_{j\neq i}\frac{\eta_{ab}t^a_i\otimes t^b_j}{z_i - z_j}\right]\langle\phi_{j_1}(z_1)\cdots\phi_{j_n}(z_n)\rangle = 0 \tag{2.6}$$

2.1 Auxiliary variables

It will be convenient to introduce the following representation for the SU(2) generators [81]:

$$J^{+}=x^{2}\frac{\partial}{\partial x}-2jx\,,\qquad J^{-}=-\frac{\partial}{\partial x}\,,\qquad J^{3}=x\frac{\partial}{\partial x}-j \tag{2.7}$$

There is also a similar algebra in terms of $\bar{x}$ for the antiholomorphic part. It is easily verified that these obey the global SU(2) algebra.

We introduce primary fields, $\phi_j(x,z)$ of the affine Lie algebra. Then:

$$\begin{aligned} J^{+}(z)\phi_j(x,w) &\sim \frac{\left(x^{2}\frac{\partial}{\partial x}-2jx\right)\phi_j(x,w)}{z-w}\\ J^{-}(z)\phi_j(x,w) &\sim \frac{-\frac{\partial}{\partial x}\phi_j(x,w)}{z-w}\\ J^{3}(z)\phi_j(x,w) &\sim \frac{\left(x\frac{\partial}{\partial x}-j\right)\phi_j(x,w)}{z-w} \end{aligned} \tag{2.8}$$

The fields $\phi_j(x,z)$ are also primary with respect to the Virasoro algebra with L_0 eigenvalue:

$$h=\frac{j(j+1)}{k+2} \tag{2.9}$$

The two point functions and three point functions are fully determined using global SU(2) and conformal transformations:

$$\langle\phi_{j_1}(x_1,z_1)\phi_{j_2}(x_2,z_2)\rangle = A(j_1)\delta_{j_1j_2}x_{12}^{2j_1}z_{12}^{-2h} \tag{2.10}$$

$$\begin{aligned}\langle\phi_{j_1}(x_1,z_1)\phi_{j_2}(x_2,z_2)\phi_{j_3}(x_3,z_3)\rangle &= C(j_1,j_2,j_3)\; x_{12}^{j_1+j_2-j_3}x_{13}^{j_1+j_3-j_2}x_{23}^{j_2+j_3-j_1}\times\\ &\quad\times z_{12}^{-h_1-h_2+h_3}z_{13}^{-h_1-h_3+h_2}z_{23}^{-h_2-h_3+h_1}\end{aligned} \tag{2.11}$$

The $C(j_1,j_2,j_3)$ are the structure constants which in principle completely determine the entire theory.

For the case of the four point correlation functions of SU(2) primaries the form is determined by global conformal and SU(2) transformations up to a function of the cross ratios. Our convention is:

$$\begin{aligned}\langle\phi_{j_1}(x_1,z_1)\phi_{j_2}(x_2,z_2)\phi_{j_3}(x_3,z_3)\phi_{j_4}(x_4,z_4)\rangle &= z_{43}^{h_2+h_1-h_4-h_3}z_{42}^{-2h_2}z_{41}^{h_3+h_2-h_4-h_1}\times\\ &\quad\times z_{31}^{h_4-h_1-h_2-h_3}x_{43}^{-j_2-j_1+j_4+j_3}x_{42}^{2j_2}\times\\ &\quad\times x_{41}^{-j_3-j_2+j_4+j_1}x_{31}^{-j_4+j_1+j_2+j_3}\;F(x,z)\end{aligned} \tag{2.12}$$

Here the invariant cross ratios are:

$$x=\frac{x_{21}x_{43}}{x_{31}x_{42}}\qquad z=\frac{z_{21}z_{43}}{z_{31}z_{42}} \tag{2.13}$$

For two and three point functions the Knizhnik-Zamolodchikov equation gives us no new information. However for the four point function (2.12) using the representation (2.7) we find it becomes a partial differential equation for $F(x,z)$:

$$(k+2)\frac{\partial}{\partial z}F(x,z)=\left[\frac{\mathcal{P}}{z}+\frac{\mathcal{Q}}{z-1}\right]F(x,z) \tag{2.14}$$

Explicitly these are:

$$
\begin{aligned}
\mathcal{P} &= -x^2(1-x)\frac{\partial^2}{\partial x^2} + ((-j_1 - j_2 - j_3 + j_4 + 1)x^2 + 2j_1 x + 2j_2 x(1-x))\frac{\partial}{\partial x} \\
&\quad + 2j_2(j_1 + j_2 + j_3 - j_4)x - 2j_1 j_2 \qquad (2.15) \\
\mathcal{Q} &= -(1-x)^2 x\frac{\partial^2}{\partial x^2} - ((-j_1 - j_2 - j_3 + j_4 + 1)(1-x)^2 + 2j_3(1-x) + 2j_2 x(1-x))\frac{\partial}{\partial x} \\
&\quad + 2j_2(j_1 + j_2 + j_3 - j_4)(1-x) - 2j_2 j_3 \qquad (2.16)
\end{aligned}
$$

For a four point correlator involving a discrete representation of spin $2j \in Z^+$ we can write the general solution to the Knizhnik-Zamolodchikov equation as:

$$
F(x,z) = \tilde{F}_0(z) + x\tilde{F}_1(z) + x^2\tilde{F}_2(z) + \cdots + x^{2j}\tilde{F}_{2j}(z) \qquad (2.17)
$$

This allows one to reduce the Knizhnik-Zamolodchikov equation to a linear *ordinary* differential equation of order $2j+1$.

3. Hamiltonian reduction

When we do a quantum hamiltonian reduction of $\mathrm{SU}(2)_k$ theories, by imposing the constraint $J^+ \sim 1$, it is well known [86]–[90] that we get the $c_{k+2,1}$ models.

The central charge of the reduced theory is precisely that of the $c_{p,q}$ model with $k+2 = p/q$ (we will always take $\gcd(p,q) = 1$):

$$
\begin{aligned}
c_{p,q} &= 1 - \frac{6(p-q)^2}{pq} \\
&= 13 - 6\left(k + 2 + \frac{1}{k+2}\right) \qquad (3.1) \\
h_{r,s} &= \frac{(pr-qs)^2 - (p-q)^2}{4pq} \qquad (3.2)
\end{aligned}
$$

If we perform hamiltonian reduction of the discrete representations of SU(2) with $2j \in Z^+$ we get the conformal weights of the $h_{1,2j+1}$ fields in the $c_{p,q}$ model:

$$
h_{1,2j+1} = \frac{j(j+1)}{k+2} - j \qquad (3.3)
$$

Here we follow an elegant realisation of this reduction that allows us to perform this at the level of the correlation functions [91]–[94]. Here we shall briefly outline the procedure and do not in any way attempt to justify its origin. The rather surprising equivalence suggested is that if one takes the limit $x_i = z_i$ in the $\mathrm{SU}(2)_k$ correlators one obtains those of the reduced $c_{k+2,1}$ model. This is indeed extremely strange as z is a physical coordinate in the plane and x is an artificial coordinate introduced to describe the SU(2) structure. Such a procedure of soldering ordinary and isotopic space was originally suggested by Polyakov [95]. A quick check of the two and three point functions of $\mathrm{SU}(2)_k$ (2.10), (2.11) does indeed yield the correct form of these correlators in the reduced theory with the

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correct conformal weights (3.3). However a much more non-trivial statement is that such a simple procedure also gives the correct four-point functions. If, rather than expanding $F(x,z)$ as a power series in x as in (2.17), we expand in the alternative basis:

$$F(x,z) = F_0(z) + (x-z)F_1(z) + (x-z)^2 F_2(z) + \cdots + (x-z)^{2j} F_{2j}(z) \qquad (3.4)$$

then it was shown [91, 92, 93, 94] that the lowest component $F_0(z)$, which is the only term surviving in the limit $x \to z$, obeys the correct differential equation for the field $h_{1,2j+1} = \frac{j(j+1)}{k+2} - j$ in the $c_{k+2,1}$ model. We have explicitly checked by hand that this works in a few of the lowest order cases.

As we shall be computing conformal blocks we shall disregard overall numerical factors which only become important when considering the consistency of the entire theory. However there are sometimes subtleties in the reduction process [96] when correlators vanish as x approaches z. This can also be seen as an obstacle in reducing the Knizhnik-Zamolodchikov equation to an expression in terms of the lowest component $F_0(z)$. We did not find any such problems in all the examples studied in this paper.

This approach gives us a very efficient and practical way to study $h_{1,s}$ correlators up to a very high level. We shall discuss several examples of $h_{1,s}$ correlators the $c_{p,q}$ models in which we have found interesting sets of rational and logarithmic solutions. Everything that we say could presumably also be reinterpreted in the $\widehat{\mathrm{SU}(2)}$ theory, as solutions for $F_0(z)$ lift up to solutions for the full $F(x,z)$, but we do not attempt this here.

4. Vacuum null vector and its fermionic partners

In this section we shall comment on the vacuum null vector and we find new fermionic partner fields.

4.1 Vacuum null vector

It is known that in the $c_{p,q}$ models by studying the vacuum null vector we can learn everything about the 'minimal' sector of operators with weights $h_{r,s}$ (3.1) with $1 \le r \le q-1$, $1 \le s \le p-1$ and identifications $h_{r,s} = h_{q-r,p-s}$ [85].

For example the Ising model at $c_{3,4} = 1/2$ has a vacuum null vector given by:

$$\mathcal{N} = 9\partial^4 T + 264((\partial^2 T)T) - 186(\partial T \partial T)) - 128(T(TT)) \qquad (4.1)$$

One can easily check by using the Virasoro algebra:

$$T(z)T(w) \sim \frac{c}{2(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial T(w)}{z-w} + \cdots \qquad (4.2)$$

and the normal ordering prescription:

$$(AB)(w) = \frac{1}{2\pi i}\oint_w \frac{dz}{z-w} A(z)B(w) \qquad (4.3)$$

that this null vector is indeed a primary field of conformal weight 6:

$$T(z)\mathcal{N}(w) \sim \frac{6\mathcal{N}(w)}{(z-w)^2} + \frac{\partial \mathcal{N}(w)}{z-w} + \cdots \qquad (4.4)$$

In the irreducible theory this null vector should be set to zero in all correlation functions. In particular the zero mode of this must vanish when applied to Virasoro primary states $|h\rangle$. We know:

$$\begin{aligned} L_n \,|h\rangle &= 0 \qquad n \geq 1 \\ L_0 \,|h\rangle &= h\,|h\rangle \end{aligned} \tag{4.5}$$

and therefore one finds:

$$\mathcal{N}_0 \,|h\rangle = -4h(2h-1)(16h-1)\,|h\rangle = 0 \tag{4.6}$$

From this one easily finds the solutions $h = 0, 1/2, 1/16$ which are well known as the conformal weights of the irreducible operator content of the Ising model. In general imposing the zero modes of the $h = (p-1)(q-1)$ vacuum null vector gives us a polynomial of rank $r = \frac{1}{2}(p-1)(q-1)$. Solving this gives us precisely the r primary operators in the 'minimal' $c_{p,q}$ model [85]. Furthermore all fusion rules in this theory can, in principle, be found from such a null vector. More details can be found in [70] and references therein.

In particular if we wish to go beyond the minimal $c_{p,q}$ model and consider fields outside the region with $1 \leq r \leq q-1, 1 \leq s \leq p-1$ we must not decouple this vacuum null vector. In order to achieve this we would have to introduce a logarithmic partner for this field. In the case of the well studied $c_{p,1}$ models the vacuum null vector is at $h = 0$ implying, as is well known, that all these extended models must have a logarithmic partner for the vacuum itself.

Continuing with the example of the Ising model we can calculate the correlator with four $h_{1,2p-1} = 6$ operators and we find:

$$F(z) = \frac{1}{z^6(1-z)^6}\left(2090z^6 - 6270z^5 + 10869z^4 - 11288z^3 + 10869z^2 - 6270z + 2090\right) \tag{4.7}$$

This conformal block is easily seen to lead to a well behaved correlator invariant under all crossing symmetries. By analysing the leading singularity as $z \to 0$ we deduce that the two point function of these fields must vanish. To see that this must be true in general consider the OPE of two vacuum null vector fields of the irreducible theory having conformal weight h. This must have the form (up to normalisation):

$$\mathcal{N}(z)\mathcal{N}(w) \sim \frac{\mathcal{N}(w)}{(z-w)^h} + \cdots \tag{4.8}$$

where $\cdots$ stands for other less singular terms. There cannot be other operators in the more singular terms as these would also be vacuum null vectors, of lower conformal weight, contradicting the fact that we are considering the vacuum null vector of the *irreducible* theory.

This is of course confirmed by explicitly calculating the OPE of (4.1) with itself. However the vanishing of the two point function of $\mathcal{N}$ immediately implies that the four point function must also vanish. In order to make the four point function non-zero and realise the conformal block (4.7) we must have one insertion of the logarithmic partner in order to make the correlator non-vanishing. This has been discussed in the LCFT literature many times before (see for example [76]). We found in all cases ($c_{p,q}$ with $p, q \leq 6$) that there is indeed a single rational solution generated by the $h_{1,2p-1} = (p-1)(q-1)$ field as we expect. However as we shall see in the next section there was always two extra *non-chiral* states as well.

4.2 Non-chiral fermionic partners

In general we found that the differential equation with four $h_{1,2p-1} = (p-1)(q-1)$ operators always admitted solutions of the form:

$$\begin{aligned} F^{(1)}(z) &= R_1(z) \\ F^{(2)}(z) &= R_1(z)\ln z + R_2(z) \\ F^{(3)}(z) &= F^{(2)}(1-z) \end{aligned} \tag{4.9}$$

where $R_1(z)$ and $R_2(z)$ are *rational* functions. The first solution $F^{(1)}(z)$ is the conformal block of the four point function of the vacuum null vector, with the subtleties about insertions of a logarithmic partner, that we have just discussed. We have already commented that, as this a bosonic field, we expect it to be invariant under all crossing symmetries:

$$R_1(z) = R_1(1-z) \qquad z^{2h} R_1\left(\frac{1}{z}\right) = R_1(z) \tag{4.10}$$

The set (4.9) is clearly closed under monodromy transformations however in order to be closed under crossing symmetries we must have:

$$z^{2h} R_2\left(\frac{1}{z}\right) = -R_2(z) + \alpha R_1(z) \tag{4.11}$$

the constant α is arbitrary but we shall always redefine $F^{(2)}(z)$ by addition of $F^{(1)}(z)$ to set α to zero.

The other solutions, as we shall presently see, correspond to extra non-chiral *fermionic* operators. To see this explicitly it is interesting to consider the example of the $c_{3,2} = 0$ model. This is of great importance in the field of percolation and polymers [15, 98, 99]. The vacuum null vector in this case is the stress tensor T itself and imposing the vanishing of this in correlators gives us just the 'minimal' topological sector. Considering fields beyond this sector we must create a logarithmic partner for the stress tensor [17].

In this model we found solutions for the $h_{1,3} = 2$ conformal blocks:

$$\begin{aligned} F^{(1)}(z) &= \frac{z^2 - z + 1}{z^2(z-1)^2} \\ F^{(2)}(z) &= F_1(z)\ln(z) - \frac{(5z^5 - 5z^4 + 12z^3 + 12z^2 - 5z + 5)}{24(z-1)z^4} \\ F^{(3)}(z) &= F_2(1-z) \end{aligned} \tag{4.12}$$

Before continuing to discuss these solutions we should comment on what occurs if one instead studies the correlators of the $h_{5,1} = 2$ field. Then one finds the *same* rational block $F^{(1)}(z)$ but a slightly different solution for $R_2(z)$ in (4.9) namely:

$$R_2(z) = \frac{\left(5z^5 - 5z^4 - 16z^3 - 16z^2 - 5z + 5\right)}{32(z-1)z^4} \tag{4.13}$$

This seems to be universal and the same rational functions always appear as a subset of both solutions. In [17] a possible extension of the conformal algebra at $c = 0$ by $h = 2$ fields was studied in which there was an extra free parameter b. They found that the operators in the $c_{3,2} = 0$ Kac-table realised only two distinct values of the parameter ($b = 5/6$, $-5/8$). The appearance of these two different solutions for $R_2(z)$ seems to be related to these results.

The rational solution $F^{(1)}(z)$ forms a well behaved chiral correlator on its own and corresponds to the vacuum null vector T. It is easy to see that this is the only primary $(2,0)$ operator in the theory as the other solutions in (4.12) on their own do not lead to single-valued correlators. It is also possible to have local $(2,2)$ operators in the theory. To see what these are we combine these conformal blocks with their anti-holomorphic components into the full correlator:

$$G(z,\bar{z}) = \sum_{a,b=1}^{3} U_{a,b} F^{(a)}(x,z)\overline{F^{(b)}(x,z)} \tag{4.14}$$

To make this single-valued everywhere we find:

$$\begin{aligned} G(z,\bar{z}) = & \; U_{1,1}F^{(1)}(z)\overline{F^{(1)}(z)} + U_{1,2}\left[F^{(1)}(z)\overline{F^{(2)}(z)} + F^{(2)}(z)\overline{F^{(1)}(z)}\right] + \\ & + U_{1,3}\left[F^{(1)}(z)\overline{F^{(3)}(z)} + F^{(3)}(z)\overline{F^{(1)}(z)}\right] \end{aligned} \tag{4.15}$$

As well as the solution corresponding to the stress tensor $F^{(1)}$ we also have two other solutions which, as we have logarithms present, do not have a diagonal form.

Now consider the effect of crossing symmetries on these solutions. Under $1 \leftrightarrow 3$ we have $z \to 1 - z$ and:

$$F^{(1)} \to F^{(1)} \qquad F^{(2)} \to F^{(3)} \qquad F^{(3)} \to F^{(2)} \tag{4.16}$$

Under $1 \leftrightarrow 4$ we have $z \to \frac{1}{z}$:

$$F^{(1)} \to z^4 F^{(1)} \qquad F^{(2)} \to -z^4 F^{(2)} \qquad F^{(3)} \to z^4\left(-i\pi F^{(1)} - F^{(2)} + F^{(3)}\right) \tag{4.17}$$

We immediately see that the other two solutions are *not* invariant under all crossing symmetries. To indicate these statistics we add extra labels to these non-chiral operators. From examining the behaviour under the crossing symmetries we find that these correspond to *non-chiral fermionic* operators $\Theta^{\pm}(z,\bar{z})$. To get the correct crossing symmetries we must have:

$$\langle\Theta^+(z_1,\bar{z_1})\Theta^-(z_2,\bar{z_2})\Theta^-(z_3,\bar{z_3})\Theta^+(z_4,\bar{z_4})\rangle = |z_{13}|^{-8}|z_{24}|^{-8}\left[F^{(1)}(z)\overline{F^{(2)}(z)} + F^{(2)}(z)\overline{F^{(1)}(z)}\right]$$

$$\langle\Theta^+(z_1,\bar{z_1})\Theta^+(z_2,\bar{z_2})\Theta^-(z_3,\bar{z_3})\Theta^-(z_4,\bar{z_4})\rangle = |z_{13}|^{-8}|z_{24}|^{-8}\left[F^{(1)}(z)\overline{F^{(3)}(z)} + F^{(3)}(z)\overline{F^{(1)}(z)}\right]$$

By expanding these we see:

$$\left\langle \Theta^{\alpha}(z_1,\bar{z_1})\Theta^{\beta}(z_2,\bar{z_2})\right\rangle = 0 \qquad \alpha,\beta=\pm \tag{4.18}$$

It appears to be a general fact that all fields beyond the minimal sector of $c = 0$ theories have vanishing two point functions. It is the non-vanishing of the four point functions that gives us a non-trivial theory.

It has been conjectured that fermionic partners to the stress tensor in $c = 0$ generate a super-algebra with U(1|1) symmetry [17]. As we have seen these fields are non-chiral and so certainly cannot be generators of an affine super-algebra.

5. Triplet solutions

In general once one considers the fusion of operators from outside the minimal region of the $c_{p,q}$ models one generates an infinite number of Virasoro primary fields. However in the $c_{p,1}$ models this infinite number of fields can be rearranged into a finite number with respect to a larger algebra - $W(2, 2p-1, 2p-1, 2p-1)$. The $h = 2$ operator is the stress tensor T and the other fields $h_{3,1} = 2p-1$ are a triplet of fields W^a with an SO(3) symmetry [77].

Although this algebra was originally found by different methods it is interesting to see how they arise from the rational solutions for the conformal blocks. For example in the second member of this series, the well known $c_{1,2} = -2$ model, we find exactly three rational four point functions for the $h_{1,3}$ or the $h_{7,1}$ fields both with $h = 3$. They are:

$$\begin{aligned}
F^{(1)}_{3333}(z) &= \frac{1}{(z-1)^6} z^4\left(6-6z+z^2\right)\\
F^{(2)}_{3333}(z) &= \frac{1}{z^6(z-1)^6}\Big(2-12z+12z^2+50z^3-225z^4+468z^5-588z^6+468z^7-225z^8+\\
&\qquad +50z^9+12z^{10}-12z^{11}+2z^{12}\Big)\\
F^{(3)}_{3333}(z) &= F^{(1)}_{3333}(1-z) = \frac{1}{z^6}\left(1-9z^2+16z^3-9z^4+z^6\right)
\end{aligned} \tag{5.1}$$

Note that $F^{(2)}_{3333}(z)$ is the unique solution satisfying:

$$\begin{aligned}
F^{(2)}_{3333}(1-z) &= F^{(2)}_{3333}(z)\\
F^{(2)}_{3333}(z) &= z^{2h}F^{(2)}_{3333}\left(\frac{1}{z}\right)
\end{aligned} \tag{5.2}$$

It therefore leads to a correlator that is invariant under all exchanges of operators. $F^{(1)}(z)$ is the unique solution with no poles as $z \to 0$ whereas $F^{(3)}(z)$ is the unique solution with no poles as $z \to 1$. It is easily seen that these requirements can be met by assuming that the fields are actually a bosonic triplet of fields W^a with the following correlators:

$$\begin{aligned}
\langle W^+(0)W^+(z)W^-(1)W^-(\infty)\rangle &= F^{(1)}_{3333}(z)\\
\langle W^3(0)W^3(z)W^3(1)W^3(\infty)\rangle &= F^{(2)}_{3333}(z)\\
\langle W^+(0)W^-(z)W^-(1)W^+(\infty)\rangle &= F^{(3)}_{3333}(z)
\end{aligned} \tag{5.3}$$

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These fields are well known in $c = -2$ and are indeed a bosonic triplet as can be verified from a simple free field construction [73]. However we see that the arguments leading us to this relied only on the existence of the three rational solutions with the stated pole structure and behaviour under crossing symmetry. We shall always write our functions $F^{(i)}$ in the same notation as in this example allowing us to immediately write the correlators and deduce the triplet structure.

One therefore suspects that the same is true in general and that a triplet of rational solutions, that are closed under crossing symmetry, will lead to a triplet algebra. We have explicitly checked for $p \leq 9$ that there is indeed a triplet of rational solutions for the operators of dimension $2p-1$ (the generators of the triplet symmetry) in the $c_{p,1}$ models.

In generalising this discussion it will be useful to note that in $c_{p,1}$ we have $h_{3,1} = h_{1,4p-1}$. We will find that it is the fields $h_{1,4p-1} = (2p-1)(2q-1)$ that become the triplet fields in the general $c_{p,q}$ models.

5.1 Correlators in the $c_{1,1}$ model

The $c_{1,1} = 1$ model is a rather peculiar case and so we shall discuss it separately in this section. The $h_{1,s}$ fields come from the hamiltonian reduction of the $2j \in Z^+$ operators of an $\mathrm{SU}(2)_{-1}$ theory. The fields have weights:

$$h_{1,2j+1} = h_{2j+1,1} = j(j+1) - j = j^2 \tag{5.4}$$

In this case we find the first few fields have dimensions: $0, 1/4, 1, \ldots$ The $j \in Z^+$ fields have integer dimensions and all correlators of these fields that we studied were found to be rational functions.

In particular we found that the $h_{1,3} = 1$ fields have three rational solutions behaving exactly as in the $c = -2$ example and so we deduce:

$$\begin{aligned}\langle W^+(0)W^+(z)W^-(1)W^-(\infty)\rangle &= \frac{z^2}{(1-z)^2}\\ \langle W^3(0)W^3(z)W^3(1)W^3(\infty)\rangle &= \frac{(1-z+z^2)^2}{z^2(1-z)^2}\\ \langle W^+(0)W^-(z)W^-(1)W^+(\infty)\rangle &= \frac{(1-z)^2}{z^2}\end{aligned} \tag{5.5}$$

These correlators (5.5) are exactly those corresponding to four point functions of affine currents J^a which generate an $\mathrm{SU}(2)_1$ Kac-Moody algebra in the extended $c_{1,1} = 1$ model.

It is interesting to examine this from the point of view of hamiltonian reduction. We start with the $\mathrm{SU}(2)_{-1}$ theory with three Kac-Moody currents and the triplet of $j = 1$ fields. Note in this case there is potential confusion as the extended fields are triplets of the $\mathrm{SU}(2)_{-1}$ algebra (as they have $j = 1$) *and* also have an extended SO(3) triplet index. After hamiltonian reduction the $\mathrm{SU}(2)_{-1}$ structure is lost but the extended one remains. What is remarkable, in this example, is that the extended structure after reduction is in fact *itself* an SU(2) affine Kac-Moody algebra, this time at level $k = 1$. As the $\mathrm{SU}(2)_1$ model is one of the very simplest rational CFTs one may hope by considering the extended triplet algebra in $\mathrm{SU}(2)_{-1}$ that this model should be a relativity simple example of a rational non-unitary CFT. It is not clear if this theory involves indecomposable representations or not.

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The four point correlators for the irreducible representations were all rational functions but further fusions may yield other representations.

5.2 Correlators in the $c_{p,q}$ models

We found for every $c_{p,q}$ model (we tested $p \leq 5, q \leq 5$) that there was always exactly three rational solutions for the $h_{1,4p-1} = (2p-1)(2q-1)$ fields. Rather more non-trivially if one exchanges p and q the differential equations are of a different order but the same set of three rational solutions solves both of them. These triplets appeared to always have a bosonic nature under crossing symmetry.

As we have discussed the $c_{2,1} = c_{1,2}$ case in the previous section we shall begin with the first new example: the $c_{2,3} = 0$ theory. In the $c_{2,3} = 0$ model the solutions are given by:

$$\begin{aligned} F^{(1)} &= \frac{1}{(1-z)^{28}}\Big(\Big(357106464 - 2856851712z + 10509841628z^2 - 23573986436z^3 + \\ &\quad + 36044249670z^4 - 39790427248z^5 + 32773983814z^6 - 20529517008z^7 + \\ &\quad + 9880147186z^8 - 3667147120z^9 + 1048374600z^{10} - 229634210z^{11} + \\ &\quad + 38248769z^{12} - 4810728z^{13} + 452625z^{14} - 30294z^{15} + 1122z^{16}\Big) z^{10}\Big) \end{aligned} \tag{5.6}$$

$$\begin{aligned} F^{(2)} &= \frac{1}{z^{28}(1-z)^{28}}\Big(2244 - 60588z + 905250z^2 - 9621456z^3 + 76497538z^4 - \\ &\quad - 459268420z^5 + 2096749200z^6 - 7334294240z^7 + 19760294372z^8 - \\ &\quad - 41059034016z^9 + 65547967628z^{10} - 79580854496z^{11} + \\ &\quad + 72088499340z^{12} - 36330724836z^{13} - 200733901482z^{14} + \\ &\quad + 2212292459088z^{15} - 14422439940116z^{16} + 68562493363130z^{17} - \\ &\quad - 254028569259777z^{18} + 763908934818536z^{19} - \\ &\quad - 1917517271406737z^{20} + 4101816418782654z^{21} - \\ &\quad - 7599053781520630z^{22} + 12352604911298080z^{23} - \\ &\quad - 17809256023135980z^{24} + 22972890487011504z^{25} - \\ &\quad - 26689578674273868z^{26} + 28044134317298400z^{27} - \\ &\quad - 26689578674273868z^{28} + 22972890487011504z^{29} - \\ &\quad - 17809256023135980z^{30} + 12352604911298080z^{31} - \\ &\quad - 7599053781520630z^{32} + 4101816418782654z^{33} - \\ &\quad - 1917517271406737z^{34} + 763908934818536z^{35} - \\ &\quad - 254028569259777z^{36} + 68562493363130z^{37} - \\ &\quad - 14422439940116z^{38} + 2212292459088z^{39} - 200733901482z^{40} - \\ &\quad - 36330724836z^{41} + 72088499340z^{42} - 79580854496z^{43} + \\ &\quad + 65547967628z^{44} - 41059034016z^{45} + 19760294372z^{46} - \\ &\quad - 7334294240z^{47} + 2096749200z^{48} - 459268420z^{49} + 76497538z^{50} - \\ &\quad - 9621456z^{51} + 905250z^{52} - 60588z^{53} + 2244z^{54}\Big) \end{aligned} \tag{5.7}$$

$$F^{(3)} = \frac{1}{z^{28}}\Big(\big(1122 + 12342z + 132855z^2 + 1026528z^3 + 5156450z^4 + 17580680z^5 + 42038555z^6 + 70854550z^7 + 83500300z^8 + 70854550z^9 + 42038555z^{10} + 17580680z^{11} + 5156450z^{12} + 1026528z^{13} + 132855z^{14} + 12342z^{15} + 1122z^{16}\big)(1-z)^{10}\Big) \tag{5.8}$$

where we have again used the same conventions as before in the labelling of the $F^{(i)}$. Although the detailed form of the solutions is extremely complicated the structure is very simple. In particular their behaviour under crossing symmetry is the same as in the previous $c = -2$ example. Therefore in an exactly analogous way to the arguments used there we deduce that the triplet of dimension $h_{1,7} = 15$ chiral fields in $c_{2,3} = 0$ behave as:

$$\begin{aligned}\langle W^+(0)W^+(z)W^-(1)W^-(\infty)\rangle &= F^{(1)}(z)\\ \langle W^3(0)W^3(z)W^3(1)W^3(\infty)\rangle &= F^{(2)}(z)\\ \langle W^+(0)W^-(z)W^-(1)W^+(\infty)\rangle &= F^{(3)}(z)\end{aligned} \tag{5.9}$$

It is not yet possible to conclude that all these algebras are closed in the sense of W-algebras as it is extremely difficult to read off the operator content from the rational correlation functions. We shall see that they do indeed appear closed as the other $h_{1,s}$ operators that could potentially contribute in the singular terms of the OPE obey Fermi statistics.

6. Doublet solutions

The triplet algebra as we have found, but certainly not proved, appears in all the $c_{p,q}$ models. However we also found that if p and q were not both odd then there was also a doublet of rational solutions with fermionic behaviour generated by the $h_{1,3p-1} = (3p/2 - 1)(3q/2 - 1)$ fields. If p and q are both odd then this field does not have $2h \in Z^+$ and so cannot be a local chiral field and is parafermionic [100].

The most familiar example is the $h_{1,2}$ or $h_{5,1}$ fields at $h = 1$ in the $c_{2,1} = -2$ model where we have the rational solutions:

$$F^1(z) = 1 - \frac{1}{z^2} \tag{6.1}$$

$$F^2(z) = 1 - \frac{1}{(1-z)^2} \tag{6.2}$$

$F^{(1)}$ is now the unique solution with no poles as $z \to 1$ and we have $F^{(2)}(z) = F^{(1)}(1-z)$. By using similar crossing symmetry arguments as before we see that there are two chiral fermionic states:

$$\begin{aligned}\langle \Psi^+(0)\Psi^-(z)\Psi^-(1)\Psi^+(\infty)\rangle &= F^{(1)}(z)\\ \langle \Psi^+(0)\Psi^+(z)\Psi^-(1)\Psi^-(\infty)\rangle &= F^{(2)}(z)\end{aligned} \tag{6.3}$$

These are precisely the $h = 1$ symplectic fermion fields $\Psi^\pm(z)$ from the $c = -2$ theory:

$$\Psi^+(z)\Psi^-(w) \sim \frac{1}{(z-w)^2} \tag{6.4}$$

We can also repeat the discussion for the next case in the $c_{p,1}$ series namely $c_{3,1} = -7$ with $h_{2,1} = 7/4$. We then have conformal blocks:

$$\begin{aligned} F^{(1)}(z) &= \frac{1}{\sqrt{z(1-z)}} \frac{(z-1)^2}{z^3} \left(2z^2 + 3z + 2\right) \\ F^{(2)}(z) &= F^{(1)}(1-z) \end{aligned} \tag{6.5}$$

We see that both solutions have branch cuts in the complex plane. They are therefore not generated by a chiral algebra but instead by a parafermionic one [100]. They still have the same form as before and so the same arguments can be made to deduce that this field has in fact a doublet nature:

$$\begin{aligned} \left\langle \Psi^+(0)\Psi^-(z)\Psi^-(1)\Psi^+(\infty)\right\rangle &= F^{(1)}(z) \\ \left\langle \Psi^+(0)\Psi^+(z)\Psi^-(1)\Psi^-(\infty)\right\rangle &= F^{(2)}(z) \end{aligned} \tag{6.6}$$

In $c_{2,3} = 0$ we also find a similar doublet nature for the $h_{1,5} = 7$ operators:

$$\begin{aligned} F^1(z) &= \frac{1}{z^{12}}\Big(\Big(-22z^9 - 44z^8 - 323z^7 - 859z^6 - 1302z^5 - 1302z^4 - 859z^3 - \\ &\qquad - 323z^2 - 44z - 22\Big)(1-z)\Big) \\ F^2(z) &= \frac{1}{(1-z)^{12}}\Big(\Big(22z^9 - 242z^8 + 1467z^7 - 6200z^6 + 18475z^5 - 37854z^4 + \\ &\qquad + 51884z^3 - 45424z^2 + 22950z - 5100\Big)z\Big) \end{aligned}$$

We have checked many other $c_{p,q}$ models, with $pq \in 2Z^+$, and always found 2 rational solutions with a fermionic symmetry for the $h_{1,3p-1} = (3p/2-1)(3q/2-1)$ fields.

7. General structure

The structure of rational solutions and Bose/Fermi assignments to operators is very suggestive. It seems that when the operators had integer conformal weights an odd number of rational solutions corresponds to bosonic operators and and an even one to fermionic ones. The cases we studied all fit into the sequence $h_{1,2np-1} = (np-1)(nq-1)$ having $2n-1$ rational solutions where $n = 1, 2, 3, \ldots$ for bosonic fields and $n = 3/2, 5/2, \ldots$ for fermionic fields.

We checked explicitly in many examples that the $h_{1,2np-1}$ fields are the only chiral $h_{1,s}$ fields by searching for rational solutions of correlation functions of other operators. If this is true then we immediately see in the singular terms of the OPE of two $h_{1,4p-1}$ triplet fields we can only create $h_{1,2p-1}$ and $h_{1,4p-1}$ fields. The next possible bosonic field is at $h_{1,6p-1} > 2h_{1,4p-1}$ and so lies beyond the singular terms in the chiral OPE. Therefore the triplet algebra must close as a W-algebra with the schematic OPE:

$$h^a_{1,4p-1} \otimes h^b_{1,4p-1} = \delta^{ab}\left[h_{1,2p-1}\right] + f^{ab}_c\left[h^c_{1,4p-1}\right] \tag{7.1}$$

where δ^{ab} and f_c^{ab} are the metric and structure constants of SU(2) and $[h]$ denotes an operator and all its descendants. Recall that the $h_{1,2p-1}$ field is the vacuum null vector of the irreducible theory and so $[h_{1,2p-1}] = [1]$.

Throughout this paper we have only considered $c_{p,q}$ with $p, q \in Z^+$. The case of $c_{p,-q}$ with $c > 25$ can also be obtained from hamiltonian reduction of $\mathrm{SU}(2)_k$ with $k+2<0$. In this case all discrete representations have negative dimensions however one also observes rational functions for certain correlators.

It is clear that these structures deserve much closer investigation.

8. The moduli space of CFTs

In this section we shall analyse the approach to local logarithmic CFTs in two particular cases. The first is the well known appearance of an indecomposable representation and the second is a situation in which operators may have extended indices, We shall analyse these in the well known $c_{2,1} = -2$ model as the operator content is particularly well known.

The first correlator that we shall analyse is the original one studied by Gurarie [2] for the $h_{1,2} = -1/8$ operators:

$$\langle \mu(z_1,\bar z_1)\mu(z_2,\bar z_2)\mu(z_3,\bar z_3)\mu(z_4,\bar z_4)\rangle = |z_{13}z_{24}|^{1/2}|z(1-z)|^{1/2}G(z,\bar z) \tag{8.1}$$

We can easily find the conformal blocks and the unique single-valued combination is:

$$G(z,\bar z) = \mathcal{F}(z)\overline{\mathcal{F}(1-z)} + \mathcal{F}(1-z)\overline{\mathcal{F}(z)} \tag{8.2}$$

where $\mathcal{F}(z)$ is the hypergeometric function: ${}_2F_1(1/2,1/2;1;z)$. This leads to a correlator invariant under all exchanges of operators. It is well known that this correlator has logarithmic singularities and it is interesting to see how these emerge as $c \to -2$.

To analyse this we examine the $c_{k+2,1}$ model when k is small. The central charge is:

$$c = 13 - 6\left(k+2+\frac{1}{k+2}\right) = -2 - \frac{9k}{2} + O(k^2) \tag{8.3}$$

The first few operators in the Kac-table have dimensions:

$$\begin{aligned} h_{1,1} &= 0 \\ h_{1,2} &= \frac{3}{4(k+2)} - \frac{1}{2} = -\frac{1}{8} - \frac{3k}{16} + O(k^2) \\ h_{1,3} &= \frac{2}{k+2} - 1 = -\frac{k}{2} + O(k^2) \end{aligned} \tag{8.4}$$

At the point $k=0$ we have $h_{1,3} = h_{1,1} = 0$ but for generic values of k there is no degeneracy in the levels. We can find the general solutions for the four point function of $h_{1,2}$ operators. The full correlator is given by:

$$\langle h_{1,2}(z_1,\bar z_1)h_{1,2}(z_2,\bar z_2)h_{1,2}(z_3,\bar z_3)h_{1,2}(z_4,\bar z_4)\rangle = |z_{13}z_{24}|^{-4h}|z|^{\frac{2k+1}{k+2}}|1-z|^{\frac{1}{k+2}}G(z,\bar z)$$

where:

$$G(z,\bar{z}) = \sum_{i,j=1}^{2} U_{i,j} F_i(z) \overline{F_j(z)} \tag{8.5}$$

and the conformal blocks $F_i(z)$ are found by solving the differential equations or via the Coulomb gas approach. They are:

$$F_1(z) = \frac{\Gamma\left(\frac{k+1}{k+2}\right)\Gamma\left(\frac{k+1}{k+2}\right)}{\Gamma\left(\frac{2k+2}{k+2}\right)} {}_2F_1\left(\frac{1}{k+2}, \frac{k+1}{k+2}; \frac{2k+2}{k+2}; z\right) \tag{8.6}$$

$$F_2(z) = z^{\frac{-k}{k+2}} \frac{\Gamma\left(\frac{1-k}{k+2}\right)\Gamma\left(\frac{k+1}{k+2}\right)}{\Gamma\left(\frac{2}{k+2}\right)} {}_2F_1\left(\frac{1}{k+2}, \frac{1-k}{k+2}; \frac{2}{k+2}; z\right) \tag{8.7}$$

We have included the normalisations so that we can use standard results. The solutions $F_1(z)$ and $F_2(z)$ are respectively the conformal blocks for the contributions from the $h_{1,1}$ and $h_{1,3}$ operators respectively as can be seen from the leading powers of z. However we immediately see that these two solutions become identical in the limit as $k \to 0$.

The full correlator must of course be single-valued everywhere. Monodromy around $z = 0$ leads to the requirement that:

$$\begin{aligned} U_{1,2} e^{2\pi i \frac{k}{k+2}} &= U_{1,2} \\ U_{2,1} e^{-2\pi \frac{k}{k+2}} &= U_{2,1} \end{aligned} \tag{8.8}$$

Now for the case of generic values of k we have $\frac{k}{k+2} \notin Z$ and we conclude $U_{1,2} = U_{2,1} = 0$ and the correlator must be diagonal:

$$G(z,\bar{z}) = U_{1,1}|F_1(z)|^2 + U_{2,2}|F_2(z)|^2 \tag{8.9}$$

Now imposing the monodromy around $z = 1$ leads to the condition [97]:

$$\frac{U_{1,1}}{U_{2,2}} = \frac{\sin\pi(a+b+c)\sin\pi b}{\sin\pi a \sin\pi c} \tag{8.10}$$

where $a = \frac{-2k-1}{k+2}$, $b = c = \frac{-1}{k+2}$. Expanding this in the limit $k \to 0$ we get:

$$\frac{U_{1,1}}{U_{2,2}} = -1 + O(k^2) \tag{8.11}$$

Therefore:

$$G(z,\bar{z}) = U_{1,1}\left(|F_1(z)|^2 - |F_2(z)|^2\right) \tag{8.12}$$

The minus sign is absolutely crucial. It signifies that we have negative norm states. Logarithms can occur when these are cancelled to leading order by the positive norm states. Expanding F_1 and F_2 gives:

$$F_1(z) = \pi\mathcal{F}(z) + kC(z) \tag{8.13}$$

$$F_2(z) = \pi\mathcal{F}(z) + kD(z) \tag{8.14}$$

where:

$$C - D = \frac{\pi^2}{2}\mathcal{F}(1-z) \tag{8.15}$$

and $\mathcal{F}(z)$ is again the hypergeometric function: ${}_2F_1\left(1/2, 1/2; 1; z\right)$. In order to make the full correlator non-vanishing in the limit $k \to 0$ we will have to choose the overall normalisation of the four point function (8.12) to be $U_{1,1} \sim 1/k$. With this choice we find that we have a smooth limit as $k \to 0$.

$$\begin{aligned} G(z,\bar{z}) &= \frac{1}{k}\left((\pi\mathcal{F}+kC)\overline{(\pi\mathcal{F}+kC)} - (\pi\mathcal{F}+kD)\overline{(\pi\mathcal{F}+kD)}\right) \\ &\to \left[\mathcal{F}(z)\overline{\mathcal{F}(1-z)} + \mathcal{F}(1-z)\overline{\mathcal{F}(z)}\right] \end{aligned} \tag{8.16}$$

In this case we were able to get a smooth approach to a logarithmic correlator from a non-logarithmic one. One might be therefore tempted to think that LCFT is merely some continuous limit of ordinary CFT. However we shall soon see that this is *not* always the case.

To illustrate this we shall examine the correlator:

$$\begin{aligned} \langle h_{1,2}(z_1,\bar{z}_1)h_{1,2}(z_2,\bar{z}_2)h_{1,3}(z_3,\bar{z}_3)h_{1,3}(z_4,\bar{z}_4)\rangle &= |z_{34}|^{4h_{1,2}-4h_{1,3}}|z_{24}|^{-4h_{1,2}}|z_{13}|^{-4h_{1,2}} \times \\ &\times |z|^{\frac{2k+1}{k+2}}|1-z|^{\frac{2}{k+2}}G(z,\bar{z}) \end{aligned} \tag{8.17}$$

Evaluating this correlator for the $c = -2$ theory we get two solutions:

$$\begin{aligned} \mathcal{F}_1(z) &= (1-z)^{-1/2} \\ \mathcal{F}_2(z) &= (1-z)^{-1/2}\arctan(\sqrt{z-1}) \end{aligned} \tag{8.18}$$

The function $\arctan(\sqrt{z-1})$ has the following behaviour near $z = 0$:[1]

$$\arctan(\sqrt{z-1}) \sim -\frac{i}{2}\ln z + \text{regular} \tag{8.19}$$

Making $G(z,\bar{z})$ single valued requires no logarithmic branch cuts and therefore we have *two* possible single-valued correlators:

$$G(z,\bar{z}) = U_{1,1}\mathcal{F}_1\overline{\mathcal{F}_1} + U_{1,2}\left(\mathcal{F}_1\overline{\mathcal{F}_2} + \mathcal{F}_2\overline{\mathcal{F}_1}\right) \tag{8.20}$$

The solution with $U_{1,2} = 0$ corresponds to the correlator:

$$\langle\mu(z_1,\bar{z}_1)\mu(z_2,\bar{z}_2)\Omega(z_3,\bar{z}_3)\Omega(z_4,\bar{z}_4)\rangle = |z_{12}|^{1/2}$$

[1] This is most easily seen using:

$$\begin{aligned} \arctan(\sqrt{z-1}) = \int \frac{1}{2z\sqrt{z-1}}\,dz &= \frac{1}{2i}\int\left[\frac{1}{z} + \frac{1}{2} + \frac{3}{8}z + \cdots\right]dz \\ &= \frac{1}{2i}\ln z + \text{regular} \end{aligned}$$

The other solution with logarithmic terms corresponds to the correlator:

$$\langle \mu(z_1,\bar z_1)\mu(z_2,\bar z_2)\Theta^+(z_3,\bar z_3)\Theta^-(z_4,\bar z_4)\rangle = |z_{12}|^{1/2}\left(\arctan(\sqrt{z-1}) + \overline{\arctan(\sqrt{z-1})}\right)$$

where $\Omega(z,\bar z)$ is the normal vacuum and $\Theta^\pm$ are a specific case at $h=0$ of the more general non-chiral fermionic operators that we have already discussed.

For any value of k we can again solve to find the conformal blocks. They are:

$$F_1(z) \;=\; {}_2F_1\left(\frac{2}{k+2},\frac{k+1}{k+2};\frac{2k+2}{k+2};z\right) \tag{8.21}$$

$$F_2(z) \;=\; z^{\frac{-k}{k+2}}{}_2F_1\left(\frac{1}{k+2},\frac{2-k}{k+2};\frac{2}{k+2};z\right) \tag{8.22}$$

Again for generic values of k we must have the diagonal correlator:

$$G(z,\bar z) = U_{2,2}\left\{\frac{U_{1,1}}{U_{2,2}}|F_1|^2 + |F_2|^2\right\} \tag{8.23}$$

Now imposing monodromy around $z=1$ we find:

$$\frac{U_{1,1}}{U_{2,2}} = -\frac{2}{\pi^2k^2} - \frac{2}{\pi^2 k} + O(1) \tag{8.24}$$

We therefore see that, in order to have a well defined limit in (8.23), we must take $U_{2,2}\sim k^2$ and we then find:

$$G(z,\bar z) \to |\mathcal{F}_1(z)|^2$$

Therefore we see that in the limit of the correlators we do *not* find the second solution $\mathcal{F}_2(z)$ corresponding to operators $\Theta^\pm$. We now have a rather interesting puzzle. For $k\neq 0$ we have no degeneracy and get a unique correlator. However at the point $k=0$ we have a *choice* of two different correlators coming from the extra indicial structure of $\Theta^\pm(z,\bar z)$. The fundamental reason for this is that the moduli space of solutions to the monodromy constraints:

$$U_{1,2}e^{2\pi i\frac{k}{k+2}} = U_{1,2}$$

is not smooth as a function of k. We see in particular that the condition is trivial if $\frac{k}{k+2}\in Z \Leftrightarrow h_{1,3}-h_{1,1}\in Z$. It is exactly in the cases in which conformal dimensions differ by integers, and we may get logarithms, that the monodromy constraints break down.

This conclusion is applicable to any conformal field theory in which one has an extended multiplet structure at a certain point. The limit of the correlators is not the same as solving the theory at the limiting point. It would be particularly interesting to analyse this in the context of disordered systems which can be studied in the replica limit or using the supersymmetric approach [30]

9. Conclusion

We have investigated the structure of the $h_{1,s}$ fields in the $c_{p,q}$ models by directly studying their correlation functions. We found that the vacuum null vector of the irreducible theory can be accompanied by two extra primary non-chiral fermionic fields. We also found a chiral triplet algebra generated by $h_{1,4p-1}$ fields. For $pq\in 2Z$ we also found extra chiral fermionic structure.

We were not able to understand the appearance of this structure but in the case of the non-chiral partners to the vacuum null vector it naively comes from the fact:

$$\ln \left| \frac{1}{z} \right| = -\ln |z|$$

It is the minus sign which indicates that operators behave in a fermionic manner under crossing symmetries. We have also seen that such an extended symmetry of particular fields does not arise in the limit of correlation functions.

It would be interesting to understand many of these points more thoroughly.

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